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1978 J. Phys. A: Math. Gen. 11 1191

(http://iopscience.iop.org/0305-4470/11/6/019)

J. Phys. A: Math. Gen., Vol. 11, No. 6, 1978. Printed in Great Britain. © 1978

One-loop effective potential with anomalous moment of the electron

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Received 7 November 1977

Abstract. We investigate the one-loop effective potential in QED by adding a Pauli term in the Green function equation for the electron. A modified Weisskopf–Schwinger Lagrangian is then computed up to order α^2.

1. Introduction

In two recently published papers (Dittrich 1976, 1977) on the Weisskopf–Schwinger Lagrangian (Weisskopf 1936, Schwinger 1951), we investigated quantum mechanical corrections for the classical Maxwell Lagrangian, $\mathscr{L}^{(0)}=\frac{1}{2}(E^2-H^2)$. In addition to the one-loop correction for various types of external fields, we computed the effective Lagrangian in QED beyond the one-loop approximation by taking into account radiative corrections.

We now want to focus our attention on yet another effect that produces corrections to the classical Lagrangian. It is the aim of this paper to obtain a correction of order α^2 which stems from the fact that the electron carries an anomalous magnetic moment. Although this problem has been dealt with in the literature (O'Connell 1968) following Weisskopf's treatment of one-loop effective potentials, we want here to make direct contact with our own Green function method which has proved to be superior in any external field problem.

2. Effective Lagrangian

Our derivation of the effective Lagrangian is based on the Green function equation

$$\left(m+\frac{1}{\mathrm{i}}\gamma\partial-e\gamma A-\mu'\tfrac{1}{2}\sigma F\right)G(x,x'|A)=\delta(x-x'). \tag{2.1}$$

Here the original primitive coupling of QED, $e\gamma A$, has been replaced by $e\gamma A+\mu'\frac{1}{2}\sigma F$, where the added term is the familiar phenomenological Pauli term which couples the electron to the external field via its anomalous magnetic moment μ'. Equation (2.1) uses the following definitions:

$$\mu'=(\tfrac{1}{2}g-1)\mu_{\mathrm{B}},\qquad \tfrac{1}{2}g-1=\alpha/2\pi+\ldots,\qquad \mu_{\mathrm{B}}=e/2m.$$

We can re-write equation (2.1) in a slightly more compact form by writing

$$[m+\gamma\Pi-\mu'\tfrac{1}{2}\sigma F]G[A]=\delta, \tag{2.2}$$

where

$$\Pi_\mu=\frac{1}{\mathrm{i}}\partial_\mu-eA_\mu, \qquad \sigma_{\mu\nu}=\tfrac{1}{2}\mathrm{i}[\gamma_\mu,\gamma_\nu].$$

In the sequel we are interested primarily in solving equation (2.2) for a constant external field. This particular assumption can be incorporated in (2.2) by the replacement

$$\Pi_\mu\to\frac{1}{\mathrm{i}}\partial_\mu+\tfrac{1}{2}e(x-x')^\nu F_{\mu\nu}.$$

Making the *ansatz* $G(x,x')=(m'-\gamma\Pi)\,\Delta(x,x')$, the basic equation to be solved turns out to be

$$[m'^2-(\gamma\Pi)^2]\,\Delta(x,x')=\delta(x-x') \tag{2.3}$$

with

$$m'^2=(m-\mu'\tfrac{1}{2}\sigma F)^2 \tag{2.4}$$

and

$$-(\gamma\Pi)^2=\Pi^2-\tfrac{1}{2}e\sigma F,$$

whereby

$$\Pi^2\to-\partial^2-\tfrac{1}{4}e^2(x-x')^\mu F^2_{\mu\nu}(x-x')^\nu. \tag{2.5}$$

(m'^2-m^2) is the field-dependent change in the (rest mass)^2 of the electron:

$$m'^2-m^2=-\mu' m\sigma F+\mu'^2(\tfrac{1}{2}\sigma F)^2. \tag{2.6}$$

If we use the following notation:

$$(\tfrac{1}{2}\sigma F)^2=2(-\mathscr{F}+\gamma_5\mathscr{G}),$$

where

$$\mathscr{F}=-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}=\tfrac{1}{2}(E^2-H^2)$$

and

$$\mathscr{G}=-\tfrac{1}{4}{}^*F^{\mu\nu}F_{\mu\nu}=\boldsymbol{E}\cdot\boldsymbol{H},$$

we find under the limiting assumption $\boldsymbol{E}=0$, $\boldsymbol{H}\neq 0$:

$$(\tfrac{1}{2}\sigma F)^2=-2\mathscr{F}=H^2$$

and therefore

$$m'^2=m^2-2\mu' m\boldsymbol{\sigma}\cdot\boldsymbol{H}+\mu'^2H^2.$$

Accordingly our Green function equation reads

$$[-\partial^2+\kappa^2-\tfrac{1}{4}e^2(x-x')^\mu F^2_{\mu\nu}(x-x')^\nu]\,\Delta(x,x')=\delta(x-x') \tag{2.7}$$

with

$$\kappa^2=\kappa_0^2-2m\mu\tfrac{1}{2}\sigma F, \qquad \mu=\mu'+\mu_{\mathrm{B}}, \qquad \kappa_0^2=m^2+\mu'^2(\tfrac{1}{2}\sigma F)^2.$$

At this point it is most convenient to make contact with our earlier work on effective Lagrangians (Dittrich 1976). There we found for the one-loop effective Lagrangian

$$\mathscr{L}[H]=\frac{1}{32\pi^2}\int_0^\infty \frac{ds}{s^3}\,e^{-i\kappa_0^2 s}\,e^{-\gamma(s)}\,\mathrm{tr}_4[\exp(i2m\mu\tfrac{1}{2}\sigma Fs)], \tag{2.8}$$

where m^2 has been replaced by the field-dependent quantity $\kappa_0^2=m^2+\mu'^2H^2$. The direction of the external constant magnetic field is assumed to be along the z axis. Hence, if there is only a constant magnetic field present, $F_{12}=-F_{21}=H$, we obtain immediately

$$\mathrm{tr}_4[\exp(i2m\mu\tfrac{1}{2}\sigma F)]=4\cos(2m\mu Hs)$$

and

$$e^{-\gamma(s)}=\det\left(\frac{eFs}{\sinh(eFs)}\right)^{1/2}=\frac{eHs}{\sin(eHs)}.$$

So far we have for the effective Lagrangian

$$\mathscr{L}[H]=\frac{1}{32\pi^2}\int_0^\infty \frac{ds}{s^3}\,e^{-i\kappa_0^2 s}\,\frac{eHs}{\sin(eHs)}4\cos(2m\mu Hs)+\textsc{ct} \tag{2.9}$$

where the contact terms (CT) have to be chosen so as to produce a finite result for equation (2.9). This can be achieved by two subtractions.

We now could evaluate (2.9) using, e.g., dimensional regularisation methods. However, since we are mainly interested in the α^2-correction to the original Weisskopf–Schwinger Lagrangian, it is justified to introduce the following approximation:

$$2m\mu=e\left(\frac{\alpha}{2\pi}+1\right)\to e$$

since $\alpha/2\pi\ll 1$. With the substitution $s\to -is$ and two subtractions we find finally

$$\mathscr{L}[H]=-\frac{1}{8\pi^2}\int_0^\infty \frac{ds}{s^3}\,e^{-m^2 s}\,e^{-(\mu' H)^2 s}[eHs\coth(eHs)-1-\tfrac{1}{3}(esH)^2]. \tag{2.10}$$

3. α^2 correction to the Maxwell Lagrangian

In this section we want to extract some limiting cases of the Lagrangian as presented in (2.10). Expanding the field-dependent exponential under the proper-time integral, we obtain

$$\mathscr{L}[H]=\mathscr{L}^{(1)}[H]+\mathscr{L}^{(2)}[H],$$

where $\mathscr{L}^{(1)}[H]$ is the well known Weisskopf–Schwinger Lagrangian, i.e.

$$\mathscr{L}^{(1)}[H]=-\frac{1}{8\pi^2}\int_0^\infty \frac{ds}{s^3}\,e^{-m^2 s}[eHs\coth(eHs)-1-\tfrac{1}{3}(esH)^2] \tag{3.1}$$

and $\mathscr{L}^{(2)}[H]$ denotes the α^2 non-linear correction to the free Maxwell theory:

$$\mathscr{L}^{(2)}[H]=\frac{1}{8\pi^2}\left(\frac{\alpha}{2\pi}\right)^2\mu_B^2H^2\int_0^\infty \frac{ds}{s^2}\,e^{-m^2 s}[eHs\coth(eHs)-1]. \tag{3.2}$$

For very strong magnetic fields,

$$H \gg H_0 = \frac{m^2}{e} = 4{\cdot}4 \times 10^{13}\ \mathrm{G},$$

we have

$$\mathscr{L}^{(1)}[H] \simeq \frac{1}{24\pi^2}(eH)^2 \ln\left(\frac{eH}{m^2}\right) \tag{3.3}$$

and

$$\mathscr{L}^{(2)}[H] \simeq \frac{1}{32\pi^2}\left(\frac{\alpha}{2\pi}\right)^2 (eH)^2 \ln\left(\frac{eH}{m^2}\right). \tag{3.4}$$

With the expression for the free Maxwell Lagrangian, $\mathscr{L}^{(0)}[H] = -\frac{1}{2}H^2$, one finds the ratios

$$\frac{\mathscr{L}^{(1)}}{\mathscr{L}^{(0)}} \simeq -\frac{\alpha}{3\pi} \ln\left(\frac{eH}{m^2}\right), \qquad \frac{\mathscr{L}^{(2)}}{\mathscr{L}^{(0)}} \simeq -\frac{1}{2}\left(\frac{\alpha}{2\pi}\right)^3 \ln\left(\frac{eH}{m^2}\right).$$

It is interesting to observe that the ratio $\mathscr{L}^{(2)}/\mathscr{L}^{(1)}$ is independent of the magnitude of the magnetic field, i.e.

$$\mathscr{L}^{(2)} \simeq \tfrac{3}{16}\alpha^2 \mathscr{L}^{(1)}. \tag{3.5}$$

This result is not unlike the strong-field limit of the one-loop effective potential with radiative corrections to the electron propagator (Dittrich 1977). Here we found $\mathscr{L}^{(2)}/\mathscr{L}^{(1)} \simeq \alpha/\pi$, which again is independent of the external field.

4. Conclusion

The main goal in this paper was to derive an exact expression for the correction to the Maxwell Lagrangian using Green function methods. This was achieved up to order α^2. The correction is due to the anomalous magnetic moment of the electron. In the strong-field limit case we showed that our result is directly proportional to the Weisskopf–Schwinger Lagrangian, the proportionality constant being α^2, i.e. independent of the external magnetic field.

Acknowledgment

The author wishes to thank Professor R F O'Connell, Baton Rouge, Louisiana, for calling his attention to an earlier paper on the subject (O'Connell 1968).

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